Exercise Sheet 3

These exercises relate to the material covered in the lecture of Week 3, and possibly previous weeks' lectures and exercises. Please submit your solutions to these exercises at the beginning of the lecture of Week 4, i.e. 12:00 on 5 November 2015. Environmentally-friendly submissions by e-mail in PDF form are welcomed! The numbers in the margin indicate approximately how many points are available for each part.

Exercise 3.1. Let $(\mathcal{V}, \|\cdot\|)$ be a normed space, let $K \subseteq \mathcal{V}$ be open and convex, and let $f: K \to \mathbb{R}$ be twice continuously differentiable throughout K.

(a) Show that f is convex if and only if its second derivative (Hessian) $\mathrm{d}^2 f(x): \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is positive semi-definite for every $x \in K$. **[8]**

(b) Show that f is strictly convex if $\mathrm{d}^2 f(x)$ is positive definite for every $x \in \mathcal{V}$, and give an example to show that the converse implication is false. **[2]**

(c) Show that, if f is convex, then the derivative of f is monotone, in the sense that **[4]**

$$(\mathrm{d}f(y) - \mathrm{d}f(x))(y - x) \geq 0 \quad \text{for all } x, y \in K.$$

Exercise 3.2. Let $\|\cdot\|$ be a norm on a vector space $\mathcal{V}$, and fix $x_0 \in \mathcal{V}$.

(a) Show that the function $J: \mathcal{V} \to [0, \infty)$ defined by $J(x) := \|x - x_0\|$ is convex, and that $J(x) := \frac{1}{2}\|x - x_0\|^2$ is strictly convex if the norm is induced by an inner product. **[6]**

(b) Give an example of a norm for which $J(x) := \frac{1}{2}\|x - x_0\|^2$ is not strictly convex and show that the norm is not strictly convex. **[4]**

Exercise 3.3. Let K be a non-empty, closed, convex subset of a Hilbert space $\mathcal{H}$.

(a) Show that, for each $y \in \mathcal{H}$, there is a unique closest element $\Pi_K y$ of K to y, i.e. such that

$$\|y - \Pi_K y\| = \inf_{z \in K} \|y - z\|,$$

and so we have a well-defined 'projection onto K' function $\Pi_K: \mathcal{H} \to K$. **[6]**

(b) Prove the variational inequality that $x = \Pi_K y$ if and only if $x \in K$ and **[5]**

$$\langle x, z - x \rangle \geq \langle y, z - x \rangle \quad \text{for all } z \in K.$$

(c) Prove that Π_K is non-expansive (also known as being short, or Lipschitz continuous with Lipschitz constant 1), i.e. **[5]**

$$\|\Pi_K y_1 - \Pi_K y_2\| \leq \|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in \mathcal{H}.$$

Exercise 3.4. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, let $A: \mathcal{H} \to \mathcal{K}$ have closed range, let $Q: \mathcal{K} \to \mathcal{K}$ and $R: \mathcal{H} \to \mathcal{H}$ be self-adjoint and positive definite, let $b \in \mathcal{K}$, and let $x_0 \in \mathcal{H}$. Let

$$J(x) := \frac{1}{2}\|Ax - b\|_Q^2 + \frac{1}{2}\|x - x_0\|_R^2.$$

(a) Calculate the gradient and Hessian (second derivative) of J at $x \in \mathcal{H}$. **[2]**

(b) Show that, regardless of the initial condition in $\mathcal{H}$, Newton's method finds the minimum of J in one iteration. **[4]**

(c) Establish the normal equations for the minimisation of J, i.e. show that **[4]**

$$\hat{x} \in \operatorname*{arg\,min}_{x \in \mathcal{H}} J(x) \iff (A^* Q A + R)\hat{x} = A^* Q b + R x_0.$$

Hint: parts (a–c) are easiest if you consider a clever choice of operator from $\mathcal{H}$ into $\mathcal{K} \oplus \mathcal{H}$, and write all the derivatives in the appropriate block form.