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Exercise Sheet 8

These exercises relate to the material covered in the lecture of Week 8, and possibly previous weeks' lectures and exercises. Please submit your solutions (in German or English) to these exercises at the beginning of the lecture of Week 9, i.e. by 12:15 on 10 December 2015.

Exercise 8.1. Let μ_0 be a Gaussian probability measure on $\mathbb{R}^n$ and suppose that the potential $\Phi(u;y)$, defined for $u \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, is quadratic in u. Show that the posterior $\mathrm{d}\mu^y \propto e^{-\Phi(u;y)}\,\mathrm{d}\mu_0$ is also a Gaussian measure on $\mathbb{R}^n$. Using whatever characterization of Gaussian measures you feel most comfortable with, extend this result to a Gaussian probability measure μ_0 on a separable Banach space $\mathcal{U}$.

Exercise 8.2. This exercise relates the 'standard assumptions' (A1)–(A4) from lectures to conditions on the forward model / observation operator H, which are in many cases easier or more natural to check. Let $\Gamma \in \mathbb{R}^{m\times m}$ be symmetric and positive definite. Suppose that $H\colon \mathcal{U} \to \mathbb{R}^m$ satisfies

(a) For every $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that, for all $u \in \mathcal{U}$,

$$\|H(u)\|_{\Gamma^{-1}} \leq \exp\bigl(\varepsilon\|u\|_{\mathcal{U}}^2 + M\bigr).$$

(b) For every $r > 0$, there exists $K > 0$ such that, for all $u_1, u_2 \in \mathcal{U}$ with $\|u_1\|_{\mathcal{U}}, \|u_2\|_{\mathcal{U}} < r$,

$$\|H(u_1) - H(u_2)\|_{\Gamma^{-1}} \leq K\bigl\|u_1 - u_2\bigr\|_{\mathcal{U}}.$$

Show that $\Phi\colon \mathcal{U} \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\Phi(u;y) \coloneqq \frac{1}{2}\bigl\langle y - H(u), \Gamma^{-1}(y - H(u))\bigr\rangle$$

satisfies the standard assumptions.

Exercise 8.3. Show that, under the standard assumptions, Bayesian inverse problems are well-posed with respect to approximation of the potential (negative log-likelihood) Φ. For simplicity, assume the observed data $y \in \mathbb{R}^m$ to be fixed. Suppose that, for $N \in \mathbb{N}$, $\Phi^N\colon \mathcal{U} \to \mathbb{R}$ is an approximation to $\Phi\colon \mathcal{U} \to \mathbb{R}$, and that, for all $\varepsilon > 0$, there exists $K = K(\varepsilon) > 0$ such that

$$\bigl|\Phi^N(u) - \Phi(u)\bigr| \leq K \exp(\varepsilon\|u\|_{\mathcal{U}}^2)\psi(N),$$

where $\lim_{N\to\infty} \psi(N) = 0$, and suppose that assumptions (A1)–(A4) hold for Φ^N and Φ with constants uniform in N. Let μ^N and μ denote the Bayesian posteriors for u given y, arrived at using Φ^N and Φ respectively. Show that there is a constant $C \geq 0$, independent of N, such that

$$d_{\mathrm{H}}(\mu^N, \mu) \leq C\psi(N),$$

where, as usual, d_{H} denotes the Hellinger metric.