FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 13

These exercises concern the lectures of Week 15, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 16, i.e. 10:00 on 2 February 2017. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 13.1. Recall that we say that X is **exponentially distributed** with rate parameter $\lambda > 0$, $X \sim \mathrm{Exponential}(\lambda)$, if

$$\rho_X(x) := \lambda e^{-\lambda x} \mathbb{I}_{(0,\infty)}(x).$$

(a) Let $X \sim \mathrm{Exponential}(\lambda)$. Given that $\mathbb{E}[X] = 1/\lambda$, find $\mathbb{V}[X]$. **[1]**

(b) Let $X_1, X_2 \sim \mathrm{Exponential}(\lambda)$ be independent. Use the convolution formula for the PDF of a sum to show that

$$\rho_{X_1+X_2}(x) = \lambda^2 x e^{-\lambda x} \mathbb{I}_{(0,\infty)}(x)$$

and, more generally, that, if $X_1, \dots, X_n \sim \mathrm{Exponential}(\lambda)$ are independent, then **[2]**

$$\rho_{X_1+\dots+X_n}(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \mathbb{I}_{(0,\infty)}(x).$$

Exercise 13.2. Suppose that $Y \sim \mathcal{N}(m, \sigma^2)$, and $X := \exp(Y)$. Then X is said to be **log-normally distributed** with parameters $m \in \mathbb{R}$ and $\sigma^2 > 0$.

(a) Show that the PDF of X is given by **[1]**

$$\rho_X(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{|\log x - m|^2}{2\sigma^2} \right) \mathbb{I}_{(0,\infty)}(x).$$

(b) Explain without detailed calculation why we should expect $\mathbb{E}[X] > e^m$. **[1]**

(c) For $k \in \mathbb{N}$, show that

$$\mathbb{E}[X^k] = \exp(km + k\sigma^2/2),$$

and hence calculate the mean and variance of X. Hint: show that

$$\mathbb{E}[X^k] = \exp(km + k\sigma^2/2)\mathbb{E}\big[\exp(k\sigma Z - k\sigma^2/2)\big],$$

where $Z \sim \mathcal{N}(0,1)$, and show that the expected value on the right-hand side is 1. **[2]**

Exercise 13.3. Let $X \sim \mathcal{N}(m_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(m_Y, \sigma_Y^2)$ be independent. Use the convolution formula to show that $X + Y \sim \mathcal{N}(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$. **[2]**

Exercise 13.4. Let X and Y have joint PDF $\rho_{(X,Y)} \colon \mathbb{R}^2 \to [0, \infty]$ defined by

$$\rho_{(X,Y)}(x,y) := \begin{cases} c, & \text{if } x \geq 0 \text{ and } x^2 + y^2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the correct value of the normalising constant $c \in \mathbb{R}$ so that $\rho_{(X,Y)}$ is a PDF. **[1]**

(b) For each $y \in \mathbb{R}$, calculate the marginal density $\rho_Y(y)$ and, if it is defined, the conditional density $\rho_{X|Y}(\cdot\,|y)$. Describe this conditional distribution of X given $Y = y$. **[2]**