FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

Lectures: T. J. Sullivan [sullivan@zib.de]
TA 1: N. Gießing [noah.giessing@hushmail.com]
TA 2: I. Klebanov [klebanov@zib.de]
TA 3: H. C. Lie [hlie@zedat.fu-berlin.de]

Exercise Sheet 14

These exercises concern the lectures of Week 16, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 17, i.e. 10:00 on 9 February 2017. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 14.1 (Moment generating functions and Hoeffding's lemma). The **moment generating function** of a real-valued random variable X is the function $M_X \colon \mathbb{R} \to \mathbb{R}$ defined by $M_X(t) := \mathbb{E}\big[e^{tX}\big]$ whenever this expected value exists. The terminology arises because formally expanding the exponential yields

$$M_X(t) = 1 + t\mathbb{E}[X] + \frac{t^2}{2}\mathbb{E}[X^2] + \dots + \frac{t^n}{n!}\mathbb{E}[X^n] + \dots$$

so $M_X(t)$ is a power series in t with the moments of X as coefficients.

(a) Suppose that $X \sim \mathrm{Uniform}([a, b])$ follows a continuous uniform distribution. Show that **[1]**

$$M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{(b-a)t}, & \text{for } t \neq 0, \\ 1, & \text{for } t = 0. \end{cases}$$

(b) Suppose that $X \sim \mathcal{N}(m, \sigma^2)$ follows a normal distribution. Calculate M_X. **[2]**

(c) Let X be a random variable with mean zero taking values in $[a, b]$. Show that, for $t \geq 0$,

$$M_X(t) \leq \exp\left(\frac{t^2(b-a)^2}{8}\right).$$

Hint: show that $\mathbb{E}\big[\exp(tX)\big] \leq \exp(\phi(t))$ for

$$\phi(t) := \log\left(\frac{b}{b-a}e^{ta} + \frac{a}{b-a}e^{tb}\right)$$

and examine the derivatives of ϕ. **[2]**

Exercise 14.2 (Quotients and normal and Cauchy random variables).

(a) Let U and V be independent absolutely continuous random variables (so that their joint PDF $\rho_{(U,V)} \colon \mathbb{R}^2 \to [0, \infty]$ is the product of their marginal PDFs $\rho_U, \rho_V \colon \mathbb{R} \to [0, \infty]$). Let $Y := U/V$ and let $Z := V$. Write (U, V) as a function of (Y, Z) and use this change of variables to show that **[2]**

$$\rho_{(Y,Z)}(y, z) = \rho_U(yz)\rho_V(z)|z|,$$
$$\rho_Y(y) = \int_{\mathbb{R}} \rho_U(yz)\rho_V(z)|z| \, \mathrm{d}z.$$

(b) Now suppose that $U \sim \mathcal{N}(0, \sigma^2)$ and $V \sim \mathcal{N}(0, 1)$ are independent and jointly normal, i.e. (U, V) has mean $(0, 0) \in \mathbb{R}^2$ and covariance matrix $\begin{pmatrix} \sigma^2 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $U/V \sim \mathrm{Cauchy}(0, \sigma)$, i.e.

$$\rho_{U/V}(y) = \frac{1}{\sigma\pi}\frac{1}{1 + (y/\sigma)^2}$$

and hence that $\mathbb{E}[U/V]$ is undefined. **[2]**

(c) Fix $r > 0$. In the plane $\mathbb{R}^2$, draw a straight line through $(0, 0)$ and $(\cos\Theta, \sin\Theta)$ with random angle $\Theta \sim \mathrm{Uniform}([-\pi/2, \pi/2))$ from the x-axis, and let Y be the y-coordinate of the point where this line meets the line $x = r$. Show that $Y \sim \mathrm{Cauchy}(0, r)$. Hint: calculate F_Y and differentiate it. **[2]**

(d) Using a computer, generate a sequence of independent draws $Y_i \sim \mathrm{Cauchy}(0, 1)$ and plot the sample average $\bar{Y}_n := \frac{1}{n}\sum_{i=1}^{n}$. (Note that parts (b) and (c) give you two ways to sample the Cauchy distribution.) If the Cauchy distribution were to have a finite mean m, then the law of large numbers would ensure that $\bar{Y}_n \to m$ as $n \to \infty$; but what happens in your experiment? **[1]**