FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 15

These exercises concern the all the lectures of Stochastik I. Students who have scored an average percentage mark in the range $[40, 50)$ are elegible to attempt these questions in order to receive credit for the exercises portion of the course. Please submit your solutions to these exercises by 10:00 on Friday 17 March 2017 to Frau Eickers, Room 134, Arnimallee 6, 14195 Berlin; in case you are not in Berlin at this time, you can also submit solutions by post, and solutions bearing a postmark of 17 March or earlier will be accepted as valid. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 15.1. Let $\Omega := \mathbb{N}$, and let $\mathcal{C} := \{\{n\} \mid n \in \mathbb{N}\}$. Let $\mathcal{A}$ be the algebra of subsets of Ω that is generated by $\mathcal{C}$, and let $\mathcal{F}$ be the σ-algebra of subsets of Ω that is generated by $\mathcal{C}$. Describe the members of $\mathcal{A}$ and $\mathcal{F}$ in simple words, and give an example of a set that is in $\mathcal{F}$ but not in $\mathcal{A}$. **[2]**

Exercise 15.2. Suppose that a fair six-sided die is rolled once, yielding a result of X. Then, a biased coin with probability $X/6$ of coming up heads is flipped twice, independently, and yields heads Y times.

(a) Write out *two* tables, with columns $x \in \{1, 2, \dots, 6\}$ and rows $y \in \{0, 1, 2\}$. In Table 1, give the conditional probabilities $p_{Y|X}(y|x) = \mathbb{P}[Y = y|X = x]$, and, in Table 2, give the joint probabilities $p_{(X,Y)}(x, y) = \mathbb{P}[X = x, Y = y]$. **[2]**

(b) Use these tables to calculate the marginal distribution (PMF) p_Y. **[1]**

(c) Calculate also the conditional distribution (PMF) $p_{X|Y}(\cdot|1)$ of X conditional upon the observation that $Y = 1$. Verify that the tabular method gives the same result as Bayes' formula. **[2]**

Exercise 15.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical (uniform) probability space with 8 outcomes $\omega_1, \dots, \omega_8$. Consider a fair 8-sided die — i.e. an octohedron — with sides labelled with $1, \dots, 8$ spots. The side showing 1 spot is red (r), the sides showing 2 and 3 spots are green (g), the sides showing 4 and 5 spots are blue (b), and remaining sides are yellow (y). Assume that the die is rolled fairly, and let X (resp. Y) denote the number of spots (resp. colour) on the uppermost side.

(a) Write down a table with a row for each $\omega \in \Omega$. Fill out three columns showing, for each $\omega \in \Omega$, the values $X(\omega)$, $Y(\omega)$, and $\mathbb{E}[X|Y](\omega)$. Verify the law of total expectation in this special case. **[3]**

(b) Write out all the sets in the σ-algebra on Ω generated by Y, i.e.

$$\sigma(Y) := \sigma\big(\{[Y = c] \mid c \in \{r, g, b, y\}\}\big).$$

Show that a discrete random variable $Z : \Omega \to \mathbb{R}$ is $\sigma(Y)$-measurable if and only if Z is constant on each event of the form $[Y = c]$. (Informally, such a Z can be 'written as a well-defined function of Y'.) Hence show that $\mathbb{E}[X|Y]$ calculated in part (a) is $\sigma(Y)$-measurable. **[3]**

Exercise 15.4. Define $\rho_X : \mathbb{R} \to \mathbb{R}$ by

$$\rho_X(x) := \begin{cases} C + Cx, & \text{if } -1 \le x \le 0, \\ C - Cx/3, & \text{if } 0 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

where $C > 0$ is a constant.

(a) Find the value of C such that ρ_X is a properly normalised PDF. Sketch this PDF. **[2]**

(b) Let X be an absolutely continuous random variable with ρ_X as its PDF. Calculate the mean, median, and variance of X. **[3]**

(c) Determine and sketch the PDF of $|X|$. **[2]**

Exercise 15.5. A family of $n \in \mathbb{N}$ dwarves plays the following game: the first dwarf throws a rock onto a line (the real line, $\mathbb{R}$), aiming for $a \in \mathbb{R}$, but missing by a random amount $U_1 \sim \mathcal{N}(0, \sigma_1^2)$, and hence landing at a random point X_1; then, for each $k = 2, \ldots, n$, dwarf k throws a rock, aiming for X_{k-1}, but missing by a random amount $U_k \sim \mathcal{N}(0, \sigma_k^2)$, and hence landing at a random point X_k. Assume that all the U_k are mutually independent. What is the distribution of the random variable X_1? What is the distribution of the random vector $U = (U_1, \ldots, U_n)$? What is the distribution of the random variable X_n? **[3]**

Exercise 15.6. Fix $\lambda > 0$ and let $X \sim \text{Exponential}(\lambda)$. Find and sketch the PDF of X conditioned upon the event $[X \leq 2/\lambda]$, and calculate its mean and variance. **[3]**